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THE EFFECTS OF VISCOSITY AND HEAT CONDUCTIVITY ON THE TRANSMISSION OF PLANE SOUND WAVES

by John R. Knudsen

DECAY OF SPHERICAL SOUND PULSES DUE TO VISCOSITY AND HEAT CONDUCTIVITY

by Joseph B. Keller

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Introduction

The usual theory of shock wave propagation omits the effects of viscosity and heat conduction. It is to be expected, however, that these dissipative mechanisms will produce observable effects on shocks which travel great distances. Since such shocks are now of considerable importance, we have attempted to examine theoretically the effect of viscosity and heat conduction on them.

In this first investigation the shock has been assumed to be so weak that all nonlinear effects can be neglected and the usual acoustic equations applied. Under these assumptions the propagation of plane pulses has been determined by Dr. J. Knudsen, and his results constitute the first, and main, part of this report. In the second part Knudsen's results are used to find the decay of a spherical pulse. This latter result may apply to spherical shocks resulting from explosions, at great distances from the explosion, when the shock becomes weak.

J. B. Keller

The Effects of Viscosity and Heat Conductivity on the Transmission of Plane Sound Pulses

John R. Knudsen

Part I. General Discussion

1. The Statement of the Problem

In connection with the problem of transmission of sound waves in a compressible fluid it is of interest to study the effects of viscosity and heat conductivity, since these effects are always present in any real situation. The purpose here is to study these effects in the case of an ideal, polytropic gas in the semi-infinite channel of Fig. 1. We assume, for the purpose of simplifying the mathematical problem, that the gas is initially at rest in the channel with constant pressure, density and temperature throughout.

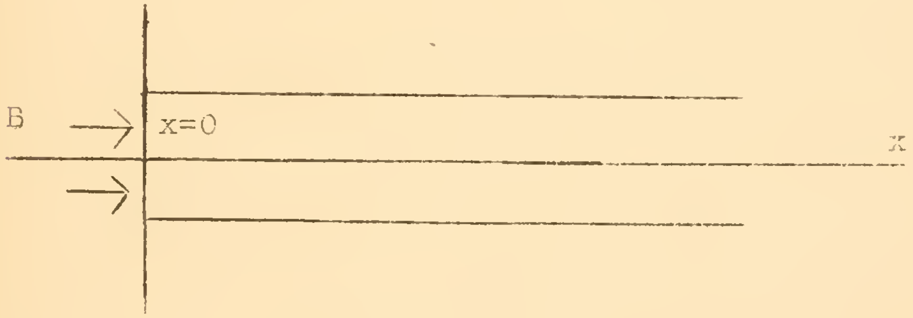

Fig. 1

Imagine that we have constructed at the left end of the channel a sort of fan or blower B directed to the right into the channel and producing the same kind of gas as is in the channel. The blower should operate in such a way that it generates at any time t a flow of gas with a prescribed velocity at x = 0. The introduction of this prescribed flow speed has the desired effect of producing a wave motion in the channel.

Assume the blower action to cause a wave to be propagated to the right into the undisturbed gas. What, then, can be said about the form and amplitude of this

wave and about the general velocity, pressure, density and temperature distribution along the channel for various values of the time t? For example, suppose that at a certain time, say t = 0, the flow speed at the mouth of the channel is suddenly raised by action of the blower from the value zero to a value, say 1, which is then maintained indefinitely. A wave front will then be propagated to the right moving into the original undisturbed gas ahead of it and followed by a region of disturbed gas behind it. In the rather trivial first approximation in which the differential equations of motion are linearized and in which viscosity and heat conductivity effects are neglected the wave assumes the form of a so-called "square wave" moving with the speed of sound into the gas at rest and followed by a steady state region with constant flow speed 1. If viscosity and heat conductivity effects are not neglected the form of the wave is distorted and the region behind it is not a steady state. It will be the purpose here to investigate in a linear or "acoustic" approximation the effects of viscosity and heat conductivity on such a wave motion.

One other simplifying feature is introduced, i.e., the boundary layer effects along the walls of the channel will be neglected. Thus the problem is one-dimensional.

Of some interest is a description of the flow as seen by an observer located at a fixed point $x = x_o$ for large values of time t. This description will show the limit flow approached in the channel behind the wave front as time increases indefinitely and the manner in which this limit is approached.

Of greater interest is the description of the wave front as seen by an observer moving with it. Accordingly we introduce an observer moving along the channel at a speed equal to the sound speed of the gas at rest, that is, an observer moving according to the law $x = x_o + c_o t$. From the point of view of such an observer the velocity and pressure

of the gas in the wave will depend upon μ and λ, the coefficients of viscosity and heat conductivity, respectively, and upon the time t. In particular, we shall be concerned with the appearance of the velocity and pressure as a functions of x_0 for large values of t.

An even better description of the wave is obtained by an observer moving according to the law $x = kc_0\sqrt{t} + c_0 t$. Inasmuch as a fixed k interval will cover an increasingly large portion of the x-axis such an observer acquires more information concerning the wave. This particular observer is suggested by the mathematical analysis of the problem which is relatively simple in this case.

As far as the nature of the prescribed velocity at the left end of the channel is concerned we shall first set up the problem in a general way with an arbitrary prescribed boundary condition at $x = 0$. Then we shall discuss the special case mentioned above in which, with the gas initially at rest in the channel, the velocity at $x = 0$ is suddenly increased at time $t = 0$ from zero to some value, say 1, which remains constant for all time. Also, as a more interesting case we take that in which the velocity at $x = 0$ is increased from zero to 1 at time $t = 0$ and then suddenly decreased to zero again at some time $t = t_1$. We are interested, in particular, in the decay of the wave in the latter case as t increases indefinitely.

With respect to values of μ and λ three situations will be discussed. These are the two extreme situations in which viscosity and heat conductivity in turn do not appear and an intermediate case in which $\alpha = \frac{\mu}{\lambda} = \frac{1}{\gamma}$ where γ is the gas constant. (When necessary to give γ a numerical value we assume the gas to be air and set $\gamma = 1.4$.) The cases $\mu = 0$ and $\lambda = 0$, although artificial from a physical standpoint present interesting mathematical problems whose conclusions at least approximate physical situations. The particular value $\frac{1}{\gamma}$ for $\frac{\mu}{\lambda}$ as an intermediary case is chosen for two

reasons. For one thing it serves to greatly simplify the mathematical analysis of the problem. Also, for $\gamma = 1.4$, it is quite close to the ratio $\frac{\mu}{\lambda} = \frac{1}{2\gamma - 1}$ which Becker has shown to be a good approximation in a real physical situation.

2. General Results

In part II will be found the mathematical solution of the problem for the cases $\alpha = \frac{1}{\gamma}$, $\alpha = 0$ and $\alpha = \infty$ with the two types of waves described above as viewed by the several observers for large values of t. We state at this point some of the more significant results.

Let u, p and ρ, all functions of x and t, be the velocity, pressure and density, respectively, of the gas in the channel. In the acoustic approximation we let $u = \delta u$, $p = p_0 + \delta p$ and $\rho = \rho_0 + \delta\rho$ and set up a linearized problem in the small quantities δu, δp, and $\delta\rho$. The quantities p_0 and ρ_0 are the constant pressure and density of the gas in its initial state of rest.

2-1. The case $\alpha = \frac{1}{\gamma}$

Consider first the wave motion resulting when the flow speed at $x = 0$ is increased at time $t = 0$ from zero to 1 and maintained at that constant value for all time $t > 0$.* As the resulting wave front moves into the undisturbed gas it leaves behind it an unsteady state. The asymptotic expression for δu obtained by an observer at $x = x_0$ for large values of t is given by

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* Actually, since δu is small for all x and t the boundary value $\delta u(0,t)$ should be small. Because of the linearization of the problem δu will at all points of the channel be proportional to $\delta u(0,t)$ and we may consider the ratio $\delta u(x,t)/\delta u(0,t)$. For convenience and with no loss of generality we choose $\delta u(0,t) = 1$. δu then is essentially the ratio of flow speed to the prescribed flow speed at the entrance to the channel and is dimensionless.

$$(2.01) \qquad \delta u \sim 1 + \frac{(\gamma-1)x_o\sqrt{T_\lambda}}{2c_o\sqrt{\gamma\pi}\;t^{3/2}} + \dots \quad .$$

c_o is the sound speed of the gas at rest and $T_\lambda = \frac{\lambda}{\rho_o c_o^2}$ is proportional to λ (and to μ since $\mu = \frac{\lambda}{\gamma}$) and has the dimension of time. The second term in the expansion is positive indicating that behind the wave front a flow speed greater than that introduced at the mouth of the channel is attained. For increasingly large values of t the flow speed decreases approaching for all observers $x = x_o$ the steady state limit flow with $\delta u = 1$. Clearly, the smaller the value of T_λ the more rapidly is this limit flow approached. For $T_\lambda = 0$ the situation reduces to that described earlier when viscosity and heat conductivity effects are neglected.

An observer moving according to the law $x = x_o + c_o t$ observes the flow speed in the wave front. For large values of t this speed is given by

$$(2.02) \quad \delta u(x_o + c_o t, t) \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{8-5\gamma}{8\gamma} - \frac{x_o}{c_o T_\lambda}\right)\sqrt{\frac{T_\lambda}{t}}$$

$$+ \frac{2(\gamma-1)}{\gamma(\gamma+3)}\, e^{-\frac{\gamma}{2c_o T_\lambda}\left(x_o + \frac{c_o}{2} t\right)} \sqrt{\frac{T_\lambda}{t}} + \dots$$

or, in terms of x and t,

$$(2.03) \quad \delta u(x,t) \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{8-5\gamma}{8\gamma} - \frac{x - c_o t}{c_o T_\lambda}\right)\sqrt{\frac{T_\lambda}{t}}$$

$$+ \frac{2(\gamma-1)}{\gamma(\gamma+3)}\, e^{-\frac{\gamma}{2c_o T_\lambda}\left(x - \frac{c_o}{2} t\right)} \sqrt{\frac{T_\lambda}{t}} + \dots \quad .$$

The significance of this result is best seen by considering δu as a function of $x_o = x - c_o t$ for fixed large t and

fixed T_λ. The third term, because of the exponential function, is very small for large t compared with the second term. Consider a small x_0-interval centered about $x_0 = 0$. Fig. 2 shows δu in this interval for various values of T_λ.

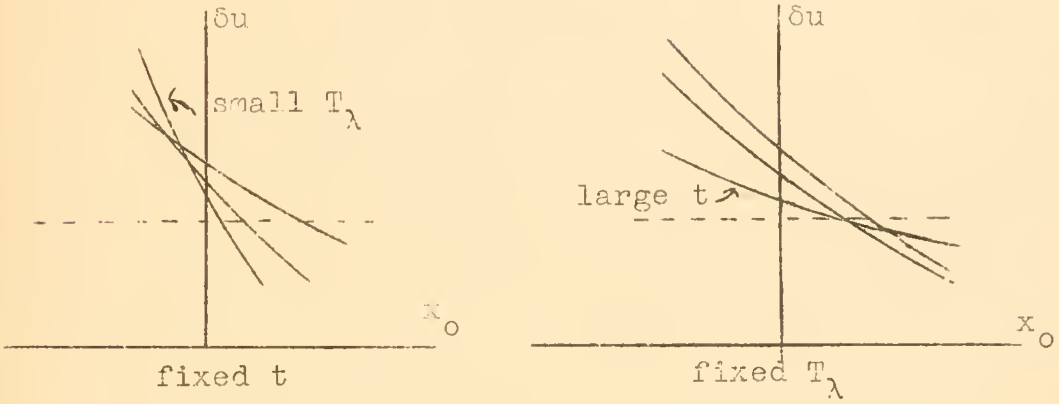

Fig. 2

For decreasing values of T_λ the x_0 interval considered must be decreased if (2.02) is to remain a good approximation for δu. As T_λ decreases the curves become steeper approaching the square wave form to be expected when viscosity and heat conductivity effects are not present. Also, for fixed values of x_0 and T_λ the velocity approaches $\frac{1}{2}$ as t increases indefinitely. This value is halfway between the zero velocity of the gas at rest ahead of the wave and the limiting value 1 of the approximately steady state flow behind the wave.

In order to stretch the x_0-interval and obtain an expression for the velocity valid over a greater range of values of $x - c_0 t$ we introduce an observer moving according to the law $x = kc_0\sqrt{t} + c_0 t$. We obtain for δu in this case the asymptotic expansion

$$(2.04)\quad \delta u(kc_o\sqrt{t}+c_ot,t) \sim \frac{1}{2}\left\{1-\operatorname{erf}\frac{k}{\sqrt{2T_\lambda}}\right\} + \frac{8-5\gamma}{16\gamma\sqrt{2\pi}}\,e^{-\frac{k^2}{2T_\lambda}}\sqrt{\frac{T_\lambda}{t}}$$

$$+\frac{4(\gamma-1)}{\gamma(\gamma+3)}\left\{1-\operatorname{erf}\frac{k}{2}\sqrt{\frac{\gamma}{T_\lambda}}\right\}\sqrt{\frac{\gamma}{\pi}}\,e^{-\frac{\gamma}{4T_\lambda}(\sqrt{t}+k)^2}\sqrt{\frac{T_\lambda}{t}}+\dots$$

or, again, in terms of x and t

$$(2.05)\quad \delta u(x,t) \sim \frac{1}{2}\left\{1-\operatorname{erf}\frac{x-c_ot}{c_o\sqrt{2T_\lambda t}}\right\} + \frac{8-5\gamma}{16\gamma\sqrt{2\pi}}\,e^{-\frac{(x-c_ot)^2}{2T_\lambda c_o^2 t}}\sqrt{\frac{T_\lambda}{t}}$$

$$+\frac{4(\gamma-1)}{\gamma(\gamma+3)}\left\{1-\operatorname{erf}\frac{1}{2}\sqrt{\frac{\gamma}{T_\lambda}}\,\frac{x-c_ot}{c_o\sqrt{t}}\right\}\sqrt{\frac{\gamma}{\pi}}\,e^{-\frac{\gamma x^2}{2T_\lambda c_o^2 t}}\sqrt{\frac{T_\lambda}{t}}+\dots\ .$$

In a small k interval centered about $k = 0$ we consider δu for large fixed t. Fig. 3 shows δu for various values of T_λ. For decreasing values of T_λ the curves become steeper approaching the square wave limit situation. For indefinitely large t the velocity δu becomes

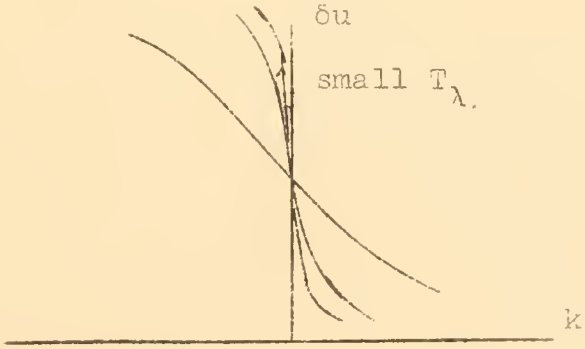

Fig. 3

$$\frac{1}{2}\left\{1-\operatorname{erf}\frac{k}{\sqrt{2T_\lambda}}\right\}$$

which in any k interval depends on T_λ. Thus more information is obtained by our present observer than by the earlier one for which the ultimate value of δu turned out to be simply $\frac{1}{2}$ over any small x_o-interval.

In the event that after a relatively small finite time t_1 the velocity at $x = 0$ is suddenly reduced to zero a second wave front is propagated to the right in the channel. It moves into the wake of the first wave front causing it to decay or die out. From the better viewpoint of the observer $x = kc_o\sqrt{t} + c_o t$ we obtain for δu the asymptotic expansion

$$(2.06)\qquad \delta u \sim \frac{t_1}{\sqrt{2\pi T_\lambda t}}\, e^{-\frac{k^2}{2T_\lambda}} + \frac{kt_1}{\sqrt{2\pi T_\lambda}}\,\frac{e^{-\frac{k^2}{2T_\lambda}}}{t}$$

$$+ \frac{4(\gamma-1)}{\gamma(\gamma+3)}\sqrt{\frac{\gamma}{\pi}}\left(1-e^{-\frac{\gamma t_1}{4T_\lambda}}\right) e^{-\frac{\gamma}{4T_\lambda}(\sqrt{t}+k)^2}\sqrt{\frac{T_\lambda}{t}}\left\{1-\operatorname{erf}\frac{k}{2}\sqrt{\frac{\gamma}{T_\lambda}}\right\} + \dots$$

or, with $k = \dfrac{x - c_o t}{c_o\sqrt{t}}$,

$$(2.07)\qquad \delta u(x,t) \sim \frac{t_1 x\, e^{-\frac{(x-c_o t)^2}{2c_o^2 T_\lambda t}}}{c_o t\sqrt{2\pi T_\lambda t}}$$

$$+ \frac{4(\gamma-1)}{\gamma(\gamma+3)}\sqrt{\frac{\gamma}{\pi}}\left(1-e^{-\frac{\gamma t_1}{4T_\lambda}}\right) e^{-\frac{\gamma x^2}{4c_o^2 T_\lambda t}}\sqrt{\frac{T_\lambda}{t}}\left\{1-\operatorname{erf}\frac{1}{2}\sqrt{\frac{\gamma}{T_\lambda}}\,\frac{x-c_o t}{c_o\sqrt{t}}\right\} + \dots\;.$$

For a fixed k interval the first term predominates and $\delta u \rightarrow 0$ as $t^{-1/2}$.

The asymptotic expansion for the pressure in this same instance is given by

$$(2.08)\quad p = p_o + \delta p$$

$$= p_o \left\{ 1 + \left[\frac{2\gamma^2 - 3\gamma + 4}{4\gamma} + \frac{2\gamma^2 + 3\gamma - 4}{4\gamma} \, \frac{x}{c_o t} \right] \frac{t_1 e^{-\frac{(x - c_o t)^2}{2c_o^2 T_\lambda t}}}{\sqrt{2\pi T_\lambda t}} \right\} + \ldots$$

whence $\delta p \longrightarrow 0$ as $t^{-1/2}$ also.

Of interest too is the flow behind the second wave front. For an observer at fixed $x = x_o$ the velocity is given by

$$(2.09)\qquad \delta u(x,t) \sim - \frac{(\gamma-1)t_1 x_o}{4c_o\sqrt{\gamma\pi}} \, \frac{\sqrt{T_\lambda}}{t^{5/2}} + \ldots \quad .$$

Thus the gas has a negative velocity in the wake of the second wave front. The limiting state of rest is approached as $t^{-5/2}$. This contrasts with the approach to the steady state with $\delta u = 1$ behind the first wave front when the second wave is not present. The approach in this case is proportional to $t^{-3/2}$.

2-2. The case $\alpha = 0$

In this somewhat artificial case in which viscosity is absent but heat conductivity present the results are similar to those of the preceding case. Ineed in determining the flow behind the second wave for fixed x and large t the result is identically that given by (2.09).

Passing over the observer $x = x_o + c_o t$ and considering only $x = kc_o\sqrt{t} + c_o t$ we observe the two wave combination to have flow speed

$$(2.10)\quad \delta u \sim \frac{t_1}{\sqrt{2\pi T_\lambda t}}\, e^{-\frac{k^2}{2T_\lambda}} - \frac{(5\gamma-1)kt_1}{8\sqrt{2\pi\gamma(\gamma-1)T_\lambda}}\, \frac{e^{-\frac{k^2}{2T_\lambda}}}{t}$$

$$+ .30\sqrt{\frac{T_\lambda}{t}}\left\{1 - \operatorname{erf} 1.02\,\frac{k}{\sqrt{T_\lambda}}\right\} e^{-\left[.43\frac{t}{T_\lambda} + .99\,\frac{k\sqrt{t}}{T_\lambda} - .75\,\frac{k^2}{T_\lambda}\right]}$$

$$\cdot\left\{\sin\left[.03\,\frac{t}{T_\lambda} + .45\,\frac{k\sqrt{t}}{T_\lambda} + .73\,\frac{k^2}{T_\lambda} + .20\right]\right.$$

$$\left. - e^{.56\,\frac{t_1}{T_\lambda}} \sin\left[.45\,\frac{t}{T_\lambda} + .45\,\frac{k\sqrt{t}}{T_\lambda} + .73\,\frac{k^2}{T_\lambda} + .20\right]\right\} .$$

Compare this with (2.06).

2-3. The case $\alpha = \infty$

In this other extreme situation of heat conductivity by no viscosity the mathematics simplifies somewhat, the differential equations being of lower order. Again the results are not unlike those of the case $\alpha = \frac{1}{\gamma}$.

For large t the observer $x = x_o + c_o t$ finds for the single pulse wave front the flow speed

$$(2.11)\qquad \delta u \sim \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{3}{8} - \frac{x_o}{c_o T_\mu}\right)\sqrt{\frac{T_\mu}{t}}$$

$(T_\mu = \frac{\mu}{\rho_o c_o^2})$ which compares with (2.03).

For the observer $x = kc_o\sqrt{t} + c_o t$ the situation is described by

$$(2.12)\qquad \delta u \sim \frac{1}{2}\left\{1 - \operatorname{erf}\frac{k}{\sqrt{2T_\mu}}\right\} + \frac{3}{8\sqrt{2\pi}}\, e^{-\frac{k^2}{2T_\mu}}\sqrt{\frac{T_\mu}{t}} + \ldots$$

which compares with (2.04).

Part II. Mathematical Analysis

3. The Differential Equations

The differential equations for a one-dimensional compressible gas flow with viscosity and heat conductivity terms present are

$$(3.01)\quad \begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + p - \mu u_x)_x = 0 \\ [\rho(\frac{1}{2}u^2 + e)]_t + [\rho u(\frac{1}{2}u^2 + i) - \mu u u_x - \lambda c_v T_\lambda]_x = 0 \end{cases}$$

where $\rho(x,t)$ is density, $u(x,t)$ velocity, $p(x,t)$ pressure, $e(x,t)$ energy, $i(x,t)$ enthalpy, $T(x,t)$ absolute temperature and c_v specific heat at constant volume.

We define, for convenience, the quantity θ, proportional to T, by $p = \rho\theta$. Empirically, we have

$$(3.02)\quad e = \frac{\theta}{\gamma - 1} \quad , \quad i = \frac{\gamma\theta}{\gamma - 1} \quad , \quad T = \frac{\theta}{c_v(\gamma - 1)}$$

and equations (3.01 become

$$(3.03)\quad \begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + \rho\theta - \mu u_x)_x = 0 \\ [\rho(\frac{1}{2}u^2 + \frac{\theta}{\gamma-1})]_t + [\rho u(\frac{1}{2}u^2 + \frac{\gamma\theta}{\gamma-1}) - \mu u u_x - \frac{\lambda\theta_x}{\gamma-1}]_x = 0 \end{cases} .$$

We introduce an acoustic approximation by setting

$$(3.04)\quad u = \delta u \quad , \quad \rho = \rho_o + \delta\rho \quad , \quad \theta = \theta_o + \delta\theta$$

where ρ_o and θ_o are the constant density and temperature of the gas at rest. Making these substitutions in (3.03) and

neglecting non-linear terms we obtain differential equations for the small quantities $\delta\rho(x,t)$, $\delta u(x,t)$ and $\delta\theta(x,t)$, i.e.,

$$(3.05)\qquad \begin{cases} (\delta\rho)_t + \rho_o(\delta u)_x = 0 \\ \rho_o(\delta u)_t + \rho_o(\delta\theta)_x + \theta_o(\delta\rho)_x = \mu(\delta u)_{xx} \\ \rho_o(\delta\theta)_t + (\gamma-1)\rho_o\theta_o(\delta u)_x = \lambda(\delta\theta)_{xx} \end{cases} .$$

4. Initial and Boundary Conditions

The differential equations (3.05) are to be solved subject to certain initial and boundary conditions. For simplicity we assume the initial state to be that of the original undisturbed gas with constant density and temperature ρ_o and θ_o. Thus

$$(4.01)\qquad \delta u(x,0) = \delta\rho(x,0) = \delta\theta(x,0) = 0 \quad .$$

At the left end of the channel we impose two boundary conditions, one on δu and one on $\delta\theta$. As it turns out these are sufficient for the evaluation of constants of integration occurring in the solution of the differential equations. The absence of $\delta\rho_{xx}$ from the differential equations seems to to indicate that no condition should be imposed on $\delta\rho$ at $x = 0$.

As the boundary condition on δu we have the prescribed velocity at the mouth of the channel due to the controlled action of the blower,

$$(4.02)\qquad \delta u(0,t) = U(t) \quad .$$

As a physically reasonable condition on the temperature we insist that there be no transfer of heat into the entrance of the channel, i.e.,

$$(4.03)\qquad \delta\theta_x(0,t) = 0 \quad .$$

As far as the right extremity of the channel, $x = \infty$, is concerned the natural conditions are

$$(4.04) \qquad \delta u(\infty,t) = \delta\rho(\infty,t) \quad , \quad \delta\theta(\infty,t) = 0 \quad .$$

The latter conditions, however, are taken care of automatically by the method of solution of the differential equations.

5. Solution of the Initial-Boundary Value Problem

The solution of equations (3.05) with initial conditions (4.01) and boundary conditions (4.02), (4.03) and (4.04) is accomplished by means of a Laplace transformation consisting of multiplying each term of equations (3.05) by e^{-st} and integrating between the limits $t = 0$ and $t = \infty$. The functions $\delta u(x,t)$, $\delta\rho(x,t)$ and $\delta\theta(x,t)$ are transformed into $w(x,s)$, $r(x,s)$ and $h(x,s)$ respectively. Equations (3.05) with the aid of the initial conditions (4.01) are transformed into

$$(5.01) \qquad \begin{cases} sr + \rho_o w_x = 0 \\ \rho_o sw + \rho_o h_x + \theta_o r_x = \mu w_{xx} \\ \rho_o sh + (\gamma-1)\rho_o\theta_o w_x = \lambda h_{xx} \end{cases}$$

and the boundary conditions (4.02) and (4.03) into

$$(5.02) \qquad w(0,s) = \overline{U}(s)$$

and

$$(5.03) \qquad h_x(0,s) = 0 \quad .$$

The transformed boundary value problem defined by (5.01), (5.02) and (5.03) can be expressed in terms of the single unknown function $w(x,s)$ by eliminating r and h. We obtain

(5.04) $\lambda(\rho_o\theta_o + \mu s)w_{xxxx} - \rho_o s(\gamma\rho_o\theta_o + \mu s + \lambda s)w_{xx} + \rho_o^2 s^3 w = 0$

as the differential equation for w(x,s) and

(5.05) $$w(0,s) = \overline{U}(s)$$

and

(5.06) $$w_{xx}(0,s) = \frac{\rho_o s^2\,\overline{U}(s)}{\rho_o\theta_o + \mu s}$$

as the boundary conditions.

The appearance of (5.04) is simplified if we write it in the form

(5.07) $$w_{xxxx} - (a^2 + b^2)w_{xx} + a^2b^2w = 0$$

where

(5.08) $$\left.\begin{matrix} a(s) \\ \\ b(s) \end{matrix}\right\}$$

$$= \sqrt{\frac{\rho_o s}{2\lambda(\rho_o\theta_o+\mu s)}\left\{\gamma\rho_o\theta_o+\mu s+\lambda s \pm \sqrt{(\gamma\rho_o\theta_o+\mu s+\lambda s)^2-4\lambda s(\rho_o\theta_o+\mu s)}\right\}}$$

in which the + sign refers to a(s) and the - sign to b(s).

The solution to (5.07) satisfying conditions (5.05) and (5.06) is

(5.09) $w(x,s)$

$$= \overline{U}(s)\left\{\frac{\dfrac{\rho_o s^2}{\rho_o\theta_o+\mu s} - b^2}{a^2 - b^2}\, e^{-a(s)x} + \frac{a^2 - \dfrac{\rho_o s^2}{\rho_o\theta_o+\mu s}}{a^2 - b^2}\, e^{-b(s)x}\right\}$$

or

(5.10) $w(x,s)$

$$= \frac{\bar{U}(s)}{2}\left\{\left[1 - \frac{\gamma\rho_o\theta_o + \mu s - \lambda s}{\sqrt{(\gamma\rho_o\theta_o + \mu s + \lambda s)^2 - 4\lambda s(\rho_o\theta_o + \mu s)}}\right]e^{-a(s)x} + \left[1 + \frac{\gamma\rho_o\theta_o + \mu s - \lambda s}{\sqrt{(\gamma\rho_o\theta_o + \mu s + \lambda s)^2 - 4\lambda s(\rho_o\theta_o + \mu s)}}\right]e^{-b(s)x}\right\} .$$

The notation is somewhat simplified if we introduce

$$(5.11) \quad T_\lambda = \frac{\lambda}{\gamma\rho_o\theta_o}\ , \quad T_\mu = \frac{\mu}{\gamma\rho_o\theta_o}\ , \quad \sigma = T_\lambda s\ , \quad \alpha = \frac{T_\mu}{T_\lambda} = \frac{\mu}{\lambda} .$$

The quantities T_λ and T_μ are proportional to λ and μ, respectively, and have the dimension of time. σ and α are dimensionless. The solution (5.10) becomes

(5.12) $w(x, \frac{\sigma}{T_\lambda})$

$$= \frac{1}{2}\,\bar{U}\left(\frac{\sigma}{T_\lambda}\right)\left\{\left[1 - \frac{1 + (\alpha-1)\sigma}{\sqrt{[1 + (\alpha+1)\sigma]^2 - \frac{4}{\gamma}\sigma(1 + \gamma\alpha\sigma)}}\right]e^{-a(\frac{\sigma}{T_\lambda})x} + \left[1 + \frac{1 + (\alpha-1)\sigma}{\sqrt{[1 + (\alpha+1)\sigma]^2 - \frac{4}{\gamma}\sigma(1 + \gamma\alpha\sigma)}}\right]e^{-b(\frac{\sigma}{T_\lambda})x}\right\}$$

where

$$(5.13) \quad \left.\begin{matrix} a(\frac{\sigma}{T_\lambda}) \\ \\ b(\frac{\sigma}{T_\lambda}) \end{matrix}\right\}$$

$$= \frac{1}{c_o T_\lambda}\sqrt{\frac{\gamma\sigma}{2(1 + \gamma\alpha\sigma)}\left\{1 + (\alpha+1)\sigma \pm \sqrt{[1 + (\alpha+1)\sigma]^2 - \frac{4}{\gamma}\sigma(1 + \gamma\alpha\sigma)}\right\}} .$$

($c_o = \sqrt{\gamma\theta_o}$ is the sound speed of the gas of rest.)

The next step is to obtain the desired velocity function $\delta u(x,t)$ from its Laplace transform $w(x, \frac{\sigma}{T_\lambda})$. This is accomplished by means of the inversion integral

$$(5.14) \qquad \delta u(x,t) = \frac{1}{2\pi i}\int_{Br} w(x, \frac{\sigma}{T_\lambda})\, e^{\frac{t}{T_\lambda}\sigma}\, \frac{d\sigma}{T_\lambda}$$

where the path of integration is the usual Bromwich line.

It is clear from (5.12) that the number and location of branch points and poles of the integrand (5.12) depend upon the value of α and upon the prescribed function whose transform is $\overline{U}(\frac{\sigma}{T_\lambda})$. Thus, in order to continue to an explicit solution we must specify α and $U(t)$. As already mentioned we will discuss three values of α, i.e., $\frac{1}{\gamma}$, 0 and ∞. We have mentioned also the two cases we shall consider for $U(t)$.

6. The Inverse Transform

The actual evaluation of (5.14) for the several cases mentioned is a lengthy process. We give here only the more significant steps.

For the case $\alpha = \frac{1}{\gamma}$ equation (5.12) becomes

$$(6.01) \quad w(x, \frac{\sigma}{T_\lambda}) = \sigma\,\overline{U}(\frac{\sigma}{T_\lambda}) \left\{ \frac{\gamma - 1}{\gamma + (\gamma - 1)\sigma}\, e^{-\frac{x}{c_o T_\lambda}\sqrt{\gamma\sigma}} + \frac{\gamma}{\sigma[\gamma + (\gamma - 1)\sigma]}\, e^{-\frac{x}{c_o T_\lambda}\frac{\sigma}{\sqrt{1+\sigma}}} \right\} .$$

(We see here the reason for the choice of $\frac{1}{\gamma}$ as a value for α. It is just the value of α that makes a perfect square of the quantity under the radical signs in (5.12). Thus it eliminates possible branch points.)

Considering $\overline{U}(\frac{\sigma}{T_\lambda})$ we have two situations. If $U(t) = 0$ for $t < 0$ and $U(t) = 1$ for $t > 0$ then $\overline{U}(\frac{\sigma}{T_\lambda}) = \frac{T_\lambda}{\sigma}$. If $U(t) = 0$ for $t < 0$ and $t > t_1$ and $U(t) = 1$ for $0 < t < t_1$ then $\overline{U}(\frac{\sigma}{T_\lambda}) = \frac{T_\lambda}{\sigma}\left(1 - e^{-\frac{t_1}{T_\lambda}\sigma}\right)$. In both cases it contributes a pole to the integrand of (5.12).

With $\overline{U}(\frac{\sigma}{T_\lambda}) = \frac{T_\lambda}{\sigma}$ (6.01) may be written as the sum of two integrals

$$(6.02) \quad I_1(x, \frac{\sigma}{T_\lambda}) = \frac{1}{2\pi i}\int_{Br} \frac{\gamma - 1}{\gamma + (\gamma-1)\sigma}\, e^{\frac{t}{T_\lambda}\sigma - \frac{x}{c_o T_\lambda}\sqrt{\gamma\sigma}}\, d\sigma$$

and

$$(6.03) \quad I_2(x, \frac{\sigma}{T_\lambda}) = \frac{1}{2\pi i}\int_{Br} \frac{\gamma}{\sigma[\gamma + (\gamma-1)\sigma]}\, e^{\frac{t}{T_\lambda}\sigma - \frac{x}{c_o T_\lambda}\frac{\sigma}{\sqrt{1+\sigma}}}\, d\sigma \, .$$

These are to be evaluated for each of three observers $x = x_o$, $x = x_o + c_o t$ and $x = k c_o\sqrt{t} + c_o t$. We cannot obtain explicit values for these integrals for all values of t. However, we can find asymptotic expansions which are valid for large values of t and this is what we are mainly interested in.

For the observer $x = x_o$ the expansions are easily obtained by choice of a proper path of integration equivalent to the Bromwich line.

For $x = x_o + c_o t$ the integral I_1 assumes the form

$$I = \frac{1}{2\pi i}\int \phi(\sigma)\, e^{tf(\sigma)}\, d\sigma$$

which by the saddle point method of steepest descents has an asymptotic expansion

$$I \sim \frac{\phi(\sigma_s)\, e^{tf(\sigma_s)}\, e^{i\beta}}{i\sqrt{2\pi}\,\sqrt{|f''(\sigma_s)|}\,\sqrt{t}}$$

where σ_s is the saddle point and β the direction of the path of integration through it.

The integral I_2 does not yield to the method of steepest descents since the saddle point σ_s of the real part of $f(\sigma)$ is a pole of the integrand. The expansion of I_2 is obtained by an appropriate deformation of the Bromwich path and reduction to standard integrals.

For the observer $x = kc_0\sqrt{t} + c_0 t$ the integral I_1 assumes the form

$$I = \frac{1}{2\pi i} \int_{Br} \phi(\sigma)\, e^{tf(\sigma) + \sqrt{t}\, g(\sigma)}\, d\sigma \quad .$$

The asymptotic expansion in this case requires a modified saddle point method based again on the saddle point of the real part of $f(\sigma)$. The expansion one obtains is

$$I \sim \frac{\phi(\sigma_s)\, e^{tf(\sigma_s) + \sqrt{t}\, g(\sigma_s) - \frac{[g'(\sigma_s)]^2}{2f''(\sigma_s)}}\, e^{i\beta}}{i\sqrt{2\pi}\,\sqrt{|f''(\sigma_s)|}\,\sqrt{t}}$$

$$\cdot \left\{ 1 - \operatorname{erf} \frac{|g'(\sigma_s)|}{2|f''(\sigma_s)|} \right\} + \dots \quad .$$

(For brevity the development of this is omitted.)

The integral I_2 again resists the saddle point method but is handled by a proper deformation of the Bromwich path.

In the case $\overline{U}(\frac{\sigma}{T_\lambda}) = \frac{T_\lambda}{\sigma}\left(1 - e^{-\frac{t_1}{T_\lambda}\sigma}\right)$ the integrals are not essentially different than in the preceding cases.

In the situation described by $\alpha = 0$ the integrand of (5.14) is somewhat more complicated than in the case $\alpha = \frac{1}{\gamma}$ because of presence of more branch points and poles. However, the saddle point methods described above suffice to evaluate the integrals.

When $\alpha = \infty$ something different occurs. The differential equation for $w(x,s)$ reduces to second order and one boundary condition has to be dropped. Rather naturally we hold on to the one prescribing the velocity of the blower and drop the one governing the heat transfer at $x = 0$.

The one integral arising in this case is, for all observers, evaluated by proper deformation of the Bromwich path and reduction to standard integrals.

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